Lecture 34. Reflections and projections via matrices

Def A set of nonzero vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m \in \mathbb{R}^n$ is orthonormal if it consists of orthogonal unit vectors.

Prop Let V be a subspace of $\mathbb{R}^n$ together with an orthonormal basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m\}$.

(1) The orthogonal projection of $\vec{x} \in \mathbb{R}^n$ onto V is

$$\boxed{\hat{x} = QQ^T\vec{x}}$$

where Q is the matrix with columns $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m$.

(2) The reflection of $\vec{x} \in \mathbb{R}^n$ through V is

$$\boxed{\tilde{x} = (2QQ^T - I)\vec{x}}$$

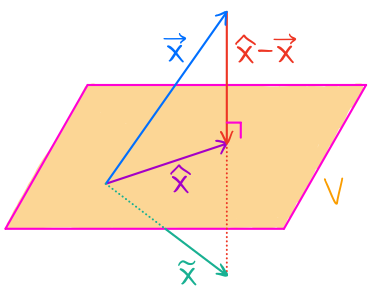
pf (1) Q^T has rows $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m$.

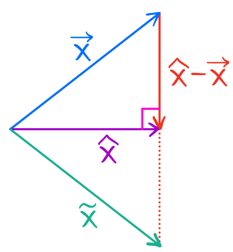
$$\Rightarrow Q^T\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ with } c_i = \vec{x} \cdot \vec{v}_i = \frac{\vec{x} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} \quad (\vec{v}_i \cdot \vec{v}_i = \|\vec{v}_i\|^2 = 1^2 = 1)$$

$$\Rightarrow QQ^T\vec{x} = Q(Q^T\vec{x}) = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \hat{x}$$

(2) $\tilde{x} = 2(\hat{x} - \vec{x}) + \vec{x} = 2\hat{x} - \vec{x}$ (cf. the last example in Lecture 32)

$$\Rightarrow \tilde{x} = 2QQ^T\vec{x} - \vec{x} = (2QQ^T - I)\vec{x}$$





Ex Find the standard matrix of each linear transformation.

(1) $T_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which reflects each vector through the line $y = 2x$.

Sol The line $y=2x$ is spanned by $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\Rightarrow \dfrac{\vec{v}}{\|\vec{v}\|} = \dfrac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ gives an orthonormal basis of the line $y=2x$.

$\Rightarrow T_1(\vec{x}) = (2QQ^T - I)\vec{x}$ where Q is the matrix with column $\dfrac{\vec{v}}{\|\vec{v}\|}$.

Hence the standard matrix is

$$2QQ^T - I = \frac{2}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}}$$

(2) $T_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which projects each vector orthogonally onto the line spanned by

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

Sol The line has an orthonormal basis given by

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

$\Rightarrow T_2(\vec{x}) = QQ^T\vec{x}$ where Q is the matrix with column $\dfrac{\vec{v}}{\|\vec{v}\|}$.

Hence the standard matrix is

$$QQ^T = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} = \boxed{\frac{1}{6} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}}$$

(3) $T_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which projects each vector orthogonally onto the plane spanned by

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

Sol $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

(neither is a multiple of the other)

$\Rightarrow$ $\vec{v}_1$ and $\vec{v}_2$ form a basis of the plane

The Gram-Schmidt process yields

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix},$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 = \vec{v}_2 - \frac{9}{9}\vec{u}_1 = \vec{v}_2 - \vec{u}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$$

$\Rightarrow$ The plane has an orthonormal basis given by

$$\frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \text{ and } \frac{\vec{u}_2}{\|\vec{u}_2\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$\Rightarrow$ $T_3(\vec{x}) = QQ^T\vec{x}$ where Q is the matrix with columns $\frac{\vec{u}_1}{\|\vec{u}_1\|}, \frac{\vec{u}_2}{\|\vec{u}_2\|}$

Hence the standard matrix is

$$QQ^T = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \boxed{\frac{1}{9}\begin{bmatrix} 8 & 2 & 2 \\ 2 & 5 & -4 \\ 2 & -4 & 5 \end{bmatrix}}$$

(4) $T_4 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which reflects each vector through the plane $x+y+z=0$.

Sol We may write the plane equation as $x=-y-z$.

With $y=s$ and $z=t$ as free variables, we find

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

$\Rightarrow$ The plane has a basis given by

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The Gram-Schmidt process yields

$$\vec{u}_1 = \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 = \vec{v}_2 - \frac{1}{2}\vec{u}_1 = \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$\Rightarrow$ The plane has an orthonormal basis given by

$$\frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \frac{\vec{u}_2}{\|\vec{u}_2\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$\Rightarrow T_4(\vec{x}) = (2QQ^T - I)\vec{x}$ where Q has columns $\frac{\vec{u}_1}{\|\vec{u}_1\|}, \frac{\vec{u}_2}{\|\vec{u}_2\|}$.

Hence the standard matrix is

$$2QQ^T - I = 2\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}}$$

<u>Note</u> We can get the same answer using the orthogonal complement of the plane, which is the line L spanned by

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The line L has an orthonormal basis given by

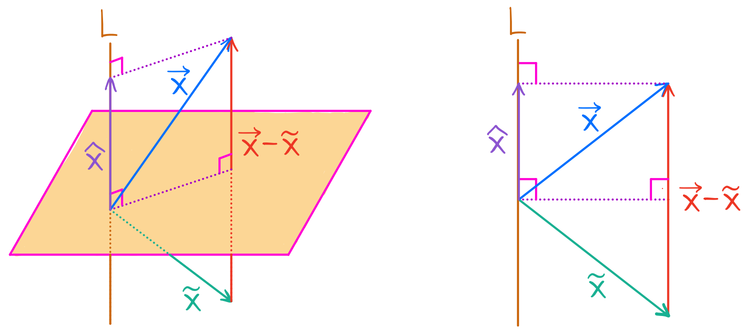
$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$\Rightarrow$ The orthogonal projection of $\vec{x} \in \mathbb{R}^n$ onto L is

$\hat{x} = QQ^T\vec{x}$ where Q is the matrix with column $\dfrac{\vec{v}}{\|\vec{v}\|}$.

For the reflection $\tilde{x} = T_4(\vec{x})$ of $\vec{x}$ through the plane, we find

$$\vec{x} - \tilde{x} = 2\hat{x}.$$



$$\Rightarrow T_4(\vec{x}) = \tilde{x} = \vec{x} - 2\hat{x} = \vec{x} - 2QQ^T\vec{x} = (I - 2QQ^T)\vec{x}$$

Hence the standard matrix is

$$I - 2QQ^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$